

COMPUTATION OF COPULAS BY FOURIER METHODS

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ABSTRACT. We provide an integral representation for (implied) copulas in terms of the moment generating function of dependent random variables. The proof uses ideas from Fourier methods for option pricing and can be applied to a large class of models from mathematical finance, including Lévy and affine processes.

1. INTRODUCTION

Copulas provide a complete characterization of the dependence between random variables, and link in a very elegant way the joint distribution with the marginal distributions via Sklar's Theorem. They are however a rather static concept and do not blend well with stochastic processes which can be used to describe the random evolution of dependent quantities, e.g. the evolution of several stock prices. Therefore other methods to create dependence in stochastic models have been developed. Multivariate stochastic processes spring immediately to mind, for example Lévy or affine processes (cf. e.g. Sato 1999, Duffie et al. 2003, Cuchiero et al. 2011), while in mathematical finance models using time-changes or linear mixture models have been developed; see e.g. Luciano and Schoutens (2006), Luciano and Semeraro (2010), Kawai (2009), Eberlein and Madan (2010) or Khanna and Madan (2009), to mention just a small part of the existing literature. In these approaches however the copula is typically not known explicitly. Another very interesting approach due to Kallsen and Tankov (2006) introduced Lévy copulas to characterize the dependence of Lévy processes.

The aim of this short note is to provide a new representation for the (implied) copula of a multidimensional random variable in terms of its moment generating function. The derivation of the main result borrows ideas from Fourier methods for option pricing, and the motivation stems from the knowledge of the moment generating function in most of the aforementioned models. This paper is organized as follows: in section 2 we provide the representation of the copula in terms of the moment generating function; the results are proved for random variables for simplicity, while stochastic processes are considered as a corollary. In section 3 we provide two examples to showcase how this method can be applied, for example, in performing sensitivity analysis of the copula with respect to the parameters of the model. Finally, section 4 concludes with some remarks.

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2. COPULAS VIA FOURIER TRANSFORM METHODS

Let \mathbb{R}^n denote the n -dimensional Euclidean space and $\langle \cdot, \cdot \rangle$ the Euclidean scalar product. The scalar product is extended from real to complex numbers as follows: for $u = (u_i)_{1 \leq i \leq n}$ and $v = (v_i)_{1 \leq i \leq n}$ in \mathbb{C}^n , set $\langle u, v \rangle := \sum_{i=1}^n u_i v_i$; therefore we do not use the Hermitian inner product $\sum_{i=1}^n u_i \bar{v}_i$. Moreover, $iv := (iv_i)_{1 \leq i \leq n}$.

Let us consider a random variable $X = (X_1, \dots, X_n)^\top \in \mathbb{R}^n$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by F the cumulative distribution function (cdf) of X and by f its probability density function (pdf). Let C denote the copula of X and c its copula density function. Analogously, let F_i and f_i denote the cdf and pdf respectively of the marginal X_i , for all $i \in \{1, \dots, n\}$. In addition, we denote by F_i^{-1} the generalized inverse of F_i , i.e. $F_i^{-1}(u) = \inf\{v \in \mathbb{R} : F_i(v) \geq u\}$.

We denote by M_X the (extended) moment generating function of X :

$$M_X(u) = \mathbb{E}[e^{\langle u, X \rangle}], \quad (2.1)$$

for all $u \in \mathbb{C}^n$ such that $M_X(u) < \infty$. In the sequel, we will assume that the following conditions are in force.

(H1): Assume that $M_X(R) < \infty$, for $R \in \mathcal{I} \subseteq \mathbb{R}^n$.

(H2): Assume that $M_X(R + i \cdot) \in L^1(\mathbb{R}^n)$ for $R \in \mathcal{J} \subseteq \mathbb{R}^n$.

(H3): Assume that $\mathcal{R} := \mathcal{I} \cap \mathcal{J} \cap \mathbb{R}_-^n \setminus \{0\} \neq \emptyset$.

Theorem 2.1. *Let X be a random variable that satisfies assumptions (H1)–(H3). The copula of X is provided by*

$$C(u) = \frac{1}{(-2\pi)^n} \int_{\mathbb{R}^n} M_X(R + iv) \frac{e^{-\langle R + iv, x \rangle}}{\prod_{i=1}^n (R_i + iv_i)} dv \Big|_{x_i = F_i^{-1}(u_i)}, \quad (2.2)$$

where $u \in [0, 1]^n$ and $R \in \mathcal{R}$.

Proof. Assumption (H2) implies that F_1, \dots, F_n are continuous and we know from Sklar's Theorem that

$$C(u_1, \dots, u_n) = F(F_1^{-1}(u_1), \dots, F_n^{-1}(u_n)); \quad (2.3)$$

see McNeil, Frey, and Embrechts (2005, Theorem 5.3) for a proof in this setting and Rüschendorf (2009) for an elegant proof in the general case.

We will evaluate the joint cdf F using the methodology of Fourier methods for option pricing. That is, we will think of the cdf as the “price” of a digital option on several fictitious assets. Let us define the function

$$g(y) = 1_{\{y_1 \leq x_1, \dots, y_n \leq x_n\}}(y), \quad x, y \in \mathbb{R}^n, \quad (2.4)$$

and denote by \hat{g} its Fourier transform. Then we have that

$$\begin{aligned} F(x) &= \mathbb{P}(X_1 \leq x_1, \dots, X_n \leq x_n) \\ &= \mathbb{E}[1_{\{X_1 \leq x_1, \dots, X_n \leq x_n\}}] \\ &= \mathbb{E}[g(X)] \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} M_X(R + iv) \hat{g}(iR - v) dv, \end{aligned} \quad (2.5)$$

where we have applied Theorem 3.2 in Eberlein, Glau, and Papapantoleon (2010). The prerequisites of this theorem are satisfied due to (H1)–(H3) and because $g_R \in L^1(\mathbb{R}^n)$, where $g_R(x) := e^{\langle R, x \rangle} g(x)$ for $R \in \mathcal{R}$.

Finally, the statement follows from (2.3) and (2.5) once we have computed the Fourier transform of g . We have for $R_i < 0$, $i \in \{1, \dots, n\}$,

$$\begin{aligned} \widehat{g}(iR - v) &= \int_{\mathbb{R}^n} e^{i\langle iR - v, y \rangle} g(y) dy \\ &= \int_{\mathbb{R}^n} e^{i\langle iR - v, y \rangle} 1_{\{y_1 \leq x_1, \dots, y_n \leq x_n\}} dy \\ &= \prod_{i=1}^n \int_{-\infty}^{x_i} e^{(-R_i - iv_i)y_i} dy_i \\ &= (-1)^n \prod_{i=1}^n \frac{e^{-(R_i + iv_i)x_i}}{(R_i + iv_i)}, \end{aligned} \quad (2.6)$$

which concludes the proof. \square

Remark 2.2. If the moment generating function of the marginals is known, the inverse function can be easily computed numerically. We have that

$$\begin{aligned} F_i^{-1}(u) &= \inf\{v \in \mathbb{R} : F_i(v) \geq u\} \\ &= \inf\{v \in \mathbb{R} : \mathbb{E}[1_{\{X_i \leq v\}}] \geq u\}, \end{aligned}$$

where the expectation can be computed using (2.5) again, while a root finding algorithm provides the infimum (using the continuity of F_i).

We can also compute the copula density function using Fourier methods, which resembles the computation of Greeks in option pricing.

Lemma 2.3. *The copula density function c of X is provided by*

$$c(u) = \frac{1}{(2\pi)^n \prod_{i=1}^n f_i(x_i)} \int_{\mathbb{R}^n} M_X(R + iv) e^{-\langle R + iv, x \rangle} dv \Big|_{x_i = F_i^{-1}(u_i)}, \quad (2.7)$$

where $u \in (0, 1)^n$ and $R \in \mathcal{R}$.

Proof. Let $u \in (0, 1)^n$, then we have that $x_i = F_i^{-1}(u_i)$ is finite for every $i \in \{1, \dots, n\}$, hence $e^{-\langle R, x \rangle}$ is bounded. Using assumption (H2) we get that the function $M_X(R + iv) e^{-\langle R + iv, x \rangle}$ is integrable and we can interchange differentiation and integration. Then we have that

$$\begin{aligned} c(u) &= \frac{\partial^n}{\partial u_1 \dots \partial u_n} C(u_1, \dots, u_n) \\ &= \frac{\partial^n}{\partial u_1 \dots \partial u_n} \frac{1}{(-2\pi)^n} \int_{\mathbb{R}^n} M_X(R + iv) \frac{e^{-\langle R + iv, x \rangle}}{\prod_{i=1}^n (R_i + iv_i)} \Big|_{x_i = F_i^{-1}(u_i)} dv \\ &= \frac{1}{(-2\pi)^n} \int_{\mathbb{R}^n} \frac{M_X(R + iv)}{\prod_{i=1}^n (R_i + iv_i)} \frac{\partial^n}{\partial u_1 \dots \partial u_n} e^{-\langle R + iv, x \rangle} \Big|_{x_i = F_i^{-1}(u_i)} dv. \end{aligned} \quad (2.8)$$

Now, using the chain rule and the inverse function theorem we get

$$\begin{aligned} \frac{\partial^n}{\partial u_1 \dots \partial u_n} \left(e^{-\langle R+iv, x \rangle} \Big|_{x_i=F_i^{-1}(u_i)} \right) \\ = (-1)^n \prod_{i=1}^n (R_i + iv_i) e^{-\langle R+iv, x \rangle} \frac{1}{\prod_{i=1}^n f_i(x_i)} \Big|_{x_i=F_i^{-1}(u_i)}, \end{aligned} \quad (2.9)$$

which combined with (2.8) yields the required result. \square

A natural application of these representations is for the calculation of copulas of random variables from multidimensional stochastic processes. There are many examples of stochastic processes where the corresponding characteristic functions are known explicitly; prominent examples are Lévy processes, self-similar additive (Sato) processes and affine processes.

Corollary 2.4. *Let $X = (X_t)_{t \geq 0}$ be an \mathbb{R}^n -valued stochastic process on a basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. Assume that the random variable X_t , $t \geq 0$, satisfies assumptions (H1)–(H3). Then the copula of X_t is provided by*

$$C_t(u) = \frac{1}{(-2\pi)^n} \int_{\mathbb{R}^n} M_{X_t}(R + iv) \frac{e^{-\langle R+iv, x \rangle}}{\prod_{i=1}^n (R_i + iv_i)} dv \Big|_{x_i=F_{X_t}^{-1}(u_i)}, \quad (2.10)$$

where $u \in [0, 1]^n$ and $R \in \mathcal{R}$. An analogous statement holds for the copula density function c_t of X_t .

3. EXAMPLES

We will demonstrate the applicability and flexibility of Fourier methods for the computation of copulas with two examples. First we consider a 2D normal random variable and next a 2D normal inverse Gaussian (NIG) Lévy process. Although the copula of the normal random variable is the well-known Gaussian copula, little was known about the copula of the NIG until recently; see Theorem 5.13 in Schmidt (2003) for a special case. Now v. Hammerstein (2011, Ch. 2) has provided a general characterization of the (implied) copula of the multidimensional NIG law using properties of normal mean-variance mixtures.

Example 3.1. The first example is simply a “sanity check” for the proposed method. We consider the 2-dimensional Gaussian distribution and compute the corresponding copula for correlation values equal to $\rho = \{-1, 0, 1\}$; see Figure 3.1 for the resulting contour plots. Of course, the copula of this example is the Gaussian copula, which for correlation coefficients equal to $\{-1, 0, 1\}$ corresponds to the *countermonotonicity* copula, the *independence* copula and the *comonotonicity* copula respectively. This is also evident from Figure 3.1.

Example 3.2. Let $X = (X_t)_{t \geq 0}$ be a 2-dimensional NIG Lévy process, i.e.

$$X_t = (X_t^1, X_t^2) \sim \text{NIG}_2(\alpha, \beta, \delta t, \mu t, \Delta), \quad t \geq 0. \quad (3.1)$$

The parameters satisfy: $\alpha, \delta \in \mathbb{R}_+$, $\beta, \mu \in \mathbb{R}^2$, and $\Delta \in \mathbb{R}^{2 \times 2}$ is a symmetric, positive definite matrix (w.l.o.g. we can assume $\det(\Delta) = 1$). Moreover,

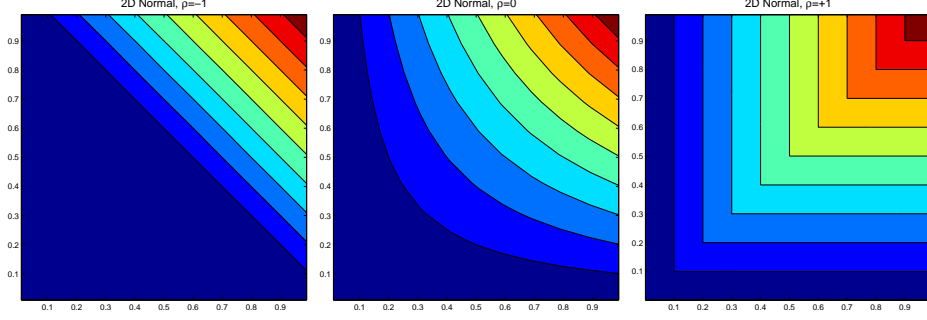


FIGURE 3.1. Contour plots of copulas for Example 3.1.

$\alpha^2 > \langle \beta, \Delta \beta \rangle$. The moment generating function of X_1 , for $u \in \mathbb{R}^2$ with $\alpha^2 - \langle \beta + u, \Delta(\beta + u) \rangle \geq 0$, is

$$M_{X_1}(u) = \exp \left(\langle u, \mu \rangle + \delta \left(\sqrt{\alpha^2 - \langle \beta, \Delta \beta \rangle} - \sqrt{\alpha^2 - \langle \beta + u, \Delta(\beta + u) \rangle} \right) \right),$$

cf. Barndorff-Nielsen (1998). The marginals are also NIG distributed and we have that $X_t^i \sim \text{NIG}(\hat{\alpha}^i, \hat{\beta}^i, \hat{\delta}^i t, \hat{\mu}^i t)$, where

$$\hat{\alpha}^i = \sqrt{\frac{\alpha^2 - \beta_j^2(\delta_{jj} - \delta_{ij}^2 \delta_{ii}^{-1})}{\delta_{ii}}}, \quad \hat{\beta}^i = \beta_i + \beta_j \delta_{ij}^2 \delta_{ii}^{-1}, \quad \hat{\delta}^i = \delta \sqrt{\delta_{ii}}, \quad \hat{\mu}^i = \mu_i,$$

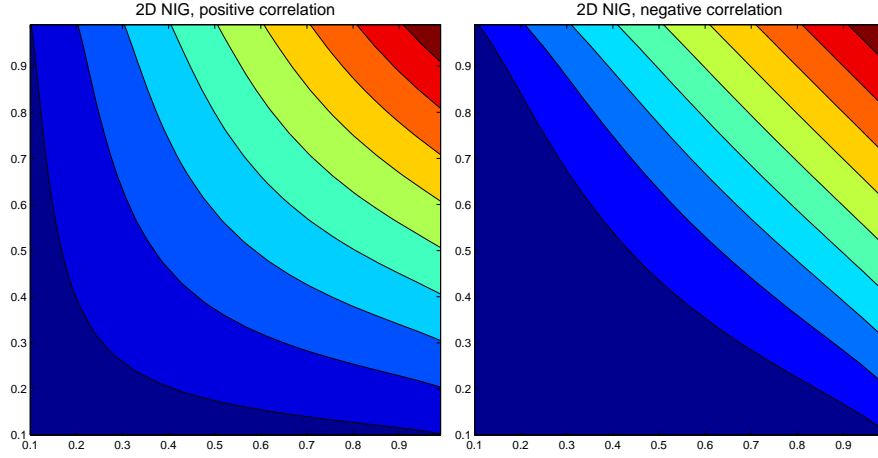
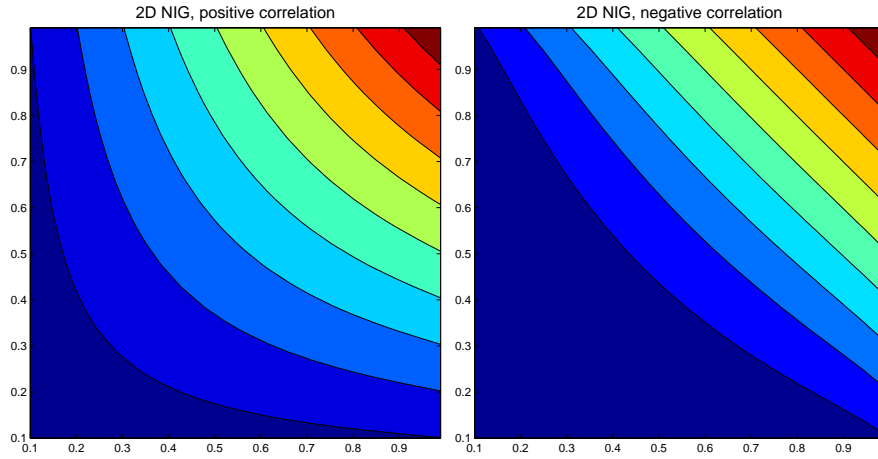
for $i = \{1, 2\}$ and $j = \{2, 1\}$; cf. e.g. Blæsild (1981, Theorem 1). Conditions (H1) and (H2) are satisfied for $R \in \mathbb{R}^2$ such that $\alpha^2 - \langle \beta + R, \Delta(\beta + R) \rangle \geq 0$, see Appendix B in Eberlein et al. (2010), thus $\mathcal{R} \neq \emptyset$.

Therefore, we can apply Theorem 2.1 to compute the copula of the NIG distribution. The parameters used in the numerical example are similar to Eberlein et al. (2010, pp. 233-234): $\alpha = 10.20$, $\beta = \begin{pmatrix} -3.80 \\ -2.50 \end{pmatrix}$, $\delta = 0.150$ and $\mu \equiv 0$, and two matrices $\Delta^+ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\Delta^- = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$, which lead to positive and negative correlation. The correlation coefficients are $\rho_+ = 0.1015$ and $\rho_- = -0.687$ respectively.

The contour plots are exhibited in Figures 3.2 and 3.3 and show clearly the influence of the different mixing matrices Δ^+ and Δ^- to the dependence structure. A minor influence of time in the dependence structure can be also observed, in particular on the examples with positive correlation.

4. FINAL REMARKS

We will not elaborate on the speed of Fourier methods compared with Monte Carlo methods in the multidimensional case; the interested reader is referred to Hurd and Zhou (2010) for a careful analysis. Moreover, Villiger (2007) provides recommendations on the efficient implementation of Fourier integrals using sparse grids in order to deal with the “curse of dimensionality”. However, let us point out that the computation of the copula function will be much quicker than the computation of the copula density, since the integrand in (2.2) decays much faster than the one in (2.7). One should think of the analogy to option prices and option Greeks again. Finally, it seems

FIGURE 3.2. Contour plots of copulas for NIG ($t = 1$).FIGURE 3.3. Contour plots of copulas for NIG ($t = 7$).

tempting to use these formulas for the computation of tail dependence coefficients. However, due to numerical instabilities at the limits, they did not yield any meaningful results.

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